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CALCULATION OF STRESS-STRAIN STATE OF ELASTIC DISC OF VARIABLE THICKNESS UNDER VIBRATION OF BLADES

The aim of the study is to calculate the stress-strain state of an elastic disc of variable thickness due to vibration of blades. For this purpose, the numerical-analytic method is developed for solving three-dimensional equations of elasticity theory. The solution is represented by a Fourier series and expansion coefficients are found from the boundary value problems for systems of ordinary differential equations along the radial coordinate. The results can be used to calculate the fatigue failure of the turbomachine disks.

Keywords: Stress-strain state, disc of variable thickness, Very high frequency loading, implicit finite difference method, Fourier series.

Introduction

In the paper, we calculate stress-strain state of gas turbine compressor disc forced by oscillations of the blades in order to study processes of Very High Cycle Fatigue (VHCF) [1]. The frequency of these oscillations is of the order of frequency of disk rotation, or a multiple of it. The development of VHCF process up to the number of cycles $N > 10^8$ may cause appearance of failure in the contact area between blades and the outer rim of the disc.

In [2,3] the stress-strain state of gas turbine compressor disc has already been calculated for Low Cycle Fatigue (LCF, $N \sim 10^5$) processes (flight cycles: take off – flight - landing) taking into account variable thickness of disk, centrifugal and aerodynamic loading as well as contact interaction of disk and blades. Aerodynamic pressures were calculated on the basis of known “isolated profile” analytical solutions for flow around plates with flow separation.

Currently there is a growing interest to the study of VHCF fracture because it becomes clear that even low amplitude vibration loads acting during long time (years) may cause the structural damage. The vibration stress amplitudes may be much less than low cycle fatigue limits and even less than yield limits. So according to classical view such weak vibrations should not be the reason of structural failure. Nevertheless the fatigue fracture happens even if vibrated structure works within elasticity limits up to failure zones appearance [1]. Therefore for fatigue life duration predictions it needs to calculate the structural stress-strain state on the basis of theory of elasticity.

It should be noted that the main disc loading is carried out in flight cycles under centrifugal and aerodynamic forces. This power background is superimposed by vibration loading due to the torsional vibration of the blades. In the adopted statement of the problem we do not consider the vibration reasons such as the effect of pressure fluctuations, the excitation of own frequency vibrations of the blades, the transitional regimes of the engine and so on, in stead we assume that the vibration parameters are known in advance. Real observational data on the amplitudes and frequencies of vibrations for disks are given in [1].

Next, we calculate stress-strain state of the disk of variable thickness due to the torsional vibrations of the blades. By the linearity of the problem this solution can be summed with known solution for LCF stress-strain state [2]. The total stress-strain states due to flight cycles and vibrations for the two extreme positions of the blade during torsional vibration are the boundaries of cyclic process and used for further assessment of fatigue life.

1. The approximate system of equations for the disc of variable thickness under periodic loads on the outer rim.

To determine the stress-strain state of this disk are solved three-dimensional equations of elasticity theory [2]. External loads on the outer rim of the disc are periodic in time and along the angular coordinate. These loads simulate the action of the torsional vibration of the blades and agreed with them in amplitude. Components of stress and strain are represented by Fourier series along the thickness and along the circumferential direction. The Fourier coefficients along the radial coordinate are obtained from system of ordinary differential equations.

In a cylindrical coordinate system r, ϑ, z, the annular disc $a \le r \le b$ has a variable thickness $2h(r)$, the thickness coordinate varies $-h(r) \le z \le h(r)$. The system of equations of the dynamic theory of elasticity [4] for a disk in a cylindrical coordinate system is:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\vartheta}}{\partial \vartheta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\vartheta\vartheta}}{r} = \rho \frac{\partial^2 u_r}{\partial t^2}$$

$$\frac{\partial \sigma_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\vartheta\vartheta}}{\partial \vartheta} + \frac{\partial \sigma_{\vartheta z}}{\partial z} + \frac{2\sigma_{r\vartheta}}{r} = \rho \frac{\partial^2 u_\vartheta}{\partial t^2}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\vartheta z}}{\partial \vartheta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = \rho \frac{\partial^2 u_z}{\partial t^2}$$

Stresses and deformations are subjected to Hooke’s law:

$$\sigma_{rr} = (\lambda + 2\mu)\varepsilon_{rr} + \lambda\varepsilon_{\vartheta\vartheta} + \lambda\varepsilon_{zz}, \quad \sigma_{\vartheta\vartheta} = \lambda\varepsilon_{rr} + (\lambda + 2\mu)\varepsilon_{\vartheta\vartheta} + \lambda\varepsilon_{zz}, \quad \sigma_{r\vartheta} = 2\mu\varepsilon_{r\vartheta}$$

$$\sigma_{zz} = \lambda\varepsilon_{rr} + \lambda\varepsilon_{\vartheta\vartheta} + (\lambda + 2\mu)\varepsilon_{zz}, \quad \sigma_{rz} = 2\mu\varepsilon_{rz}, \quad \sigma_{\vartheta z} = 2\mu\varepsilon_{\vartheta z}$$

Kinematic relations are:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\vartheta\vartheta} = \frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_r}{r}, \quad \varepsilon_{r\vartheta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \vartheta} + \frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r}\right)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad \varepsilon_{\vartheta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \vartheta} + \frac{\partial u_\vartheta}{\partial z}\right)$$

Here λ , μ are Lame's elastic moduli, ρ is the material density. Farther we use dimensionless stresses measured in $\lambda + 2\mu$, and dimensionless coordinates measured in internal rim radius *a*.

Boundary conditions at free surfaces at $z = \pm h(r)$ are:

$$\sigma_{rz} - h'\sigma_{rr} = 0, \quad \sigma_{\vartheta z} - h'\sigma_{r\vartheta} = 0, \quad \sigma_{zz} - h'\sigma_{rz} = 0$$

Due to periodicity of loading along ϑ the displacements are sought as Fourier series:

$$u_r = \sum_{n=1}^{\infty}(uz + u_3 z^3)\sin n\vartheta,$$

$$u_\vartheta = \sum_{n=0}^{\infty}(vz + v_3 z^3)\cos n\vartheta,$$

$$u_z = \sum_{n=1}^{\infty}(w + w_2 z^2 + w_4 z^4)\sin n\vartheta$$

Relevant representations for stresses are:

$$\sigma_{rr} = \sum_{n=1}^{\infty}(\sigma z + \sigma_3 z^3)\sin n\vartheta, \quad \sigma_{\vartheta\vartheta} = \sum_{n=1}^{\infty}(sz + s_3 z^3)\sin n\vartheta,$$

$$\sigma_{zz} = \sum_{n=1}^{\infty}(\Sigma z + \Sigma_3 z^3)\sin n\vartheta, \quad \sigma_{r\vartheta} = \sum_{n=0}^{\infty}(\tau z + \tau_3 z^3)\cos n\vartheta,$$

$$\sigma_{rz} = \sum_{n=1}^{\infty}(p + p_2 z^2 + p_4 z^4)\sin n\vartheta, \quad \sigma_{\vartheta z} = \sum_{n=0}^{\infty}(T + T_2 z^2 + T_4 z^4)\cos n\vartheta$$

For brevity, the index n is omitted for all coefficients included under the sign of the sum. These coefficients σ, τ , p , u , v , w are new (auxiliary) unknown functions of radial coordinate r .

We investigate stationary vibrations of the disc. All (additional to the stress-strain state of flight cycles) components of stresses, deformations and displacements vary in time according to harmonic law, for instance, for displacements as $ue^{i\omega t}$, $ve^{i\omega t}$, $we^{i\omega t}$. Substitute expressions for displacements and stresses in to equations of the theory of elasticity and equating terms of equal powers of z until z^3, as a result we obtain ordinary differential equations for the auxiliary variables at different n=0,1,2…:

$$\frac{d\sigma}{dr}=\left[\left(\frac{\lambda(1+h'^2)}{(1+\lambda)}-1\right)\frac{1}{r}-\frac{2h'}{h}\right]\sigma+\frac{n}{r}\tau+\frac{2}{h^2}p+\left[\frac{(1-\lambda)(1+2\lambda)}{(1+\lambda)}\frac{1}{r^2}-\rho\omega^2\right]u-\frac{n(1-\lambda)(1+2\lambda)}{(1+\lambda)}\frac{1}{r^2}v$$

$$\frac{d\tau}{dr}=-\frac{n}{r}\frac{\lambda(1+h'^2)}{(1+\lambda)}\sigma-2\left(\frac{1}{r}+\frac{h'}{h}\right)\tau-\frac{n}{r^2}\frac{(1-\lambda)(1+2\lambda)}{(1+\lambda)}u+\left[\frac{n}{r^2}\frac{(1-\lambda)(1+2\lambda)}{(1+\lambda)}+\frac{2\mu}{h^2}-\rho\omega^2\right]v+\frac{2\mu}{h^2}\frac{n}{r}w$$

$$\frac{dp}{dr}=-h'^2\sigma-\frac{1}{r}p+\mu\frac{n}{r}v+\left(\mu\frac{n^2}{r^2}-\rho\omega^2\right)w$$

$$\frac{du}{dr}=\frac{(1-\lambda h'^2)}{(1-\lambda^2)}\sigma-\frac{\lambda}{(1+\lambda)}\frac{1}{r}u+\frac{\lambda}{(1+\lambda)}\frac{n}{r}v \qquad (1)$$

$$\frac{dv}{dr}=\frac{1}{\mu}\tau-\frac{n}{r}u+\frac{1}{r}v$$

$$\frac{dw}{dr}=\frac{1}{\mu}p-u$$

We emphasize that this system of equations is solved separately for each harmonic n.
All other stress components $\sigma_3, s, s_3, \Sigma, \Sigma_3, \tau_3, p_{2,4}, T_{2,4}$ and displacement components $u_3, v_3, w_{2,4}$ are defined using auxiliary unknown functions σ, τ, p, u, v, w after the solution of the written system of ordinary differential equations. To obtain a closed boundary value problem for the auxiliary functions for each value of n, in the boundary conditions at $z=\pm h(r)$ we had neglected members of small order in h such as: $\Delta\Sigma = h'^2h^2\sigma_3 - h^2\Sigma_3$, $\Delta T = h'h\tau_3 - h^2T_4$, $\Delta p = h'h\sigma_3 - h^2 p_4$. Direct numerical calculations confirm the smallness of the discarded terms.

2. The boundary conditions for the torsional vibration and the solution of problem

We consider a blade as a plate of rectangular cross-section and width d. Number of blades is N_0. Action of blade on disc due to blade torsion we represent by distributed tangential load at the external rim of disc. Outside of contact zones of disc and blades these surface loads are equal to zero.

Boundary conditions at radial boundaries $r = a$ and $r = b$ with periodic (along angular coordinate) loads are:

$r = a$: $\quad u = 0, \quad v = 0, \quad w = 0$

$r = b$: $\quad \sigma = 0, \quad \tau = \tau_b(\vartheta), \quad p = p_b(\vartheta)$

where $\tau_b(\vartheta)$ and $p_b(\vartheta)$ known functions, defined below. These boundary conditions relate to fixed internal rim of the disc (r=a), and to loaded external rim of the disc (r=b). Known functions of right hand sides relate to stresses at root cross-sections of blades.

To calculate the values of $\tau_b(\vartheta)$ and $p_b(\vartheta)$ the known solution for the torsion of plate of rectangular cross section [5] is used. In Fig. 1 the distribution of shear stresses on rectangular root section of the plate for torsional twist sweatshop γ [6].

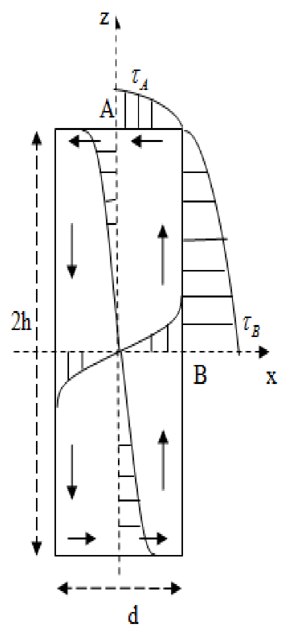


Fig 1. Distribution of shear stresses at blade root cross section.

Maximal shear stress τ_{rz} is in point B: $\tau_B = \tau_{\max} = \dfrac{K_1}{K_2}\mu\gamma d$

and in point A: $\tau_A = k\tau_B$, где $\quad K_1 \approx \dfrac{1}{3+2(d/h+d^2/h^2)}, \qquad K_2 \approx \dfrac{1}{3+1.8d/h}$.

For *d/h<<1* we have $k \approx 0.8$. Hence, $\tau_B = \mu\gamma d$, $\quad \tau_A = 0.8\tau_B$.

Exact solution for shear stresses at root cross section is represented by rather complicated expression [5]. Instead we use simplified approximation. Simplified dependencies are linear along one coordinate and quadratic along another coordinate (see Fig. 1):

$$\tau_{rx} = \tau_{r\vartheta} = -\tau_A \frac{z}{h}\left[1 - \frac{x^2}{(d/2)^2}\right], \qquad |x| \le d/2$$

$$\tau_{rz} = \tau_B \left[1 - \frac{z^2}{h^2}\right]\frac{x}{(d/2)}, \qquad |z| \le h$$

These relations are taken as approximate values of boundary shear stresses τ_{rz} and τ_{rx}. Using substitution $\vartheta = x/b$, $|\vartheta| \le \delta$, $\delta = d/(2b) << 1$, the right hand pert functions of the boundary conditions at $r = b$ can be written as:

$$\tau_b(\vartheta) = Q_0\left(1 - \vartheta^2/\delta^2\right), \qquad Q_0 = -0.8\mu\gamma d/h, \qquad |\vartheta| \le \delta$$

$$p_b(\vartheta) = T_0\, \vartheta/\delta, \qquad T_0 = \mu\gamma d, \qquad |\vartheta| \le \delta$$

Periodic boundary loads can be represented by Fourier series (one period $-\pi/N_0 < \vartheta < \pi/N_0$):

$$\tau_b(\vartheta) = \sum_{k=0}^{\infty} \tau^{(k)} \cos\left(kN_0\vartheta\right),$$

$$\tau^{(0)} = 2Q_0 N_0 \delta/(3\pi), \quad \tau^{(k)} = \frac{4Q_0}{\pi k^2 N_0 \delta}\left(\frac{\sin(kN_0\delta)}{kN_0\delta} - \cos(kN_0\delta)\right) \tag{2}$$

By analogy axial shear stresses also are represented by Fourier series:

$$p_b(\vartheta) = \sum_{k=1}^{\infty} p^{(k)} \sin\left(kN_0\vartheta\right),$$

$$p^{(k)} = \frac{2}{k\pi} T_0 \left(\frac{\sin(kN_0\delta)}{kN_0\delta} - \cos(kN_0\delta)\right) \tag{3}$$

So for different $n = kN_0$, *k=0,1,2…* it needs to solve the system of equations (1) with boundary values $\tau^{(k)}$ and $p^{(k)}$ at $r = b$.

The system of ordinary differential equations (1) with boundary conditions (2) and (3) are solved numerically by using finite difference implicit scheme [7].

After that stress components are defined by summation of Fourier series for $n = kN_0$, *k=0,1,2…* The number of Fourier series terms for practical convergence does not exceed 20.

3. Numerical results.

The shape of titanium disk cross section is shown in Fig. 2. The initial data are: $a = 0.05$m, $b = 0.4$m, $d = 0.01$m, γ=0.1 rad/m, $\omega = 628$ *1/s*, $\lambda = 78$ *GPa*, $\mu = 44$ *GPa*, $\rho = 4370$ *kg/m*^{3}.

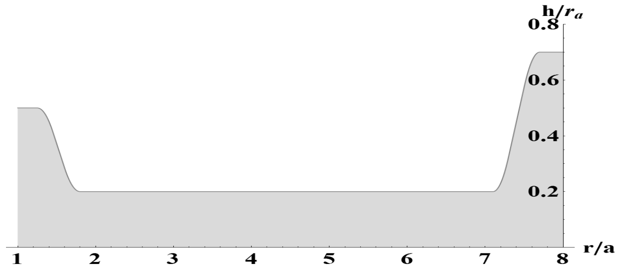


Fig. 2. Cross section of the disk.

In Fig. 3 the radial distributions of stress components are shown for cross section $\vartheta = 0.5$ (under the blade). The stresses decay rapidly with distance from the outer rim of the disc.

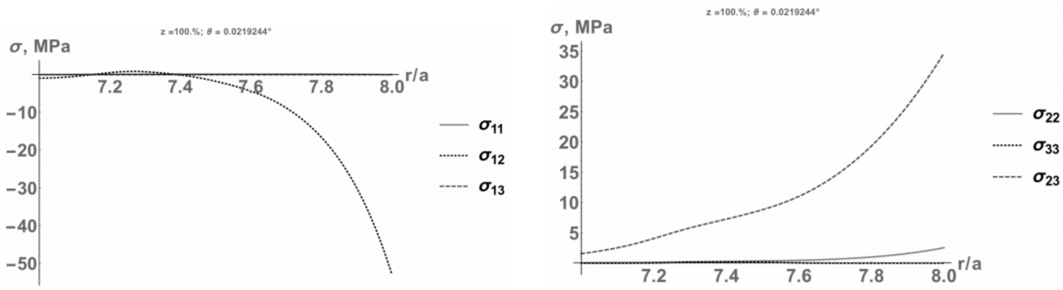


Fig. 3. Radial distributions of stresses under the blade.

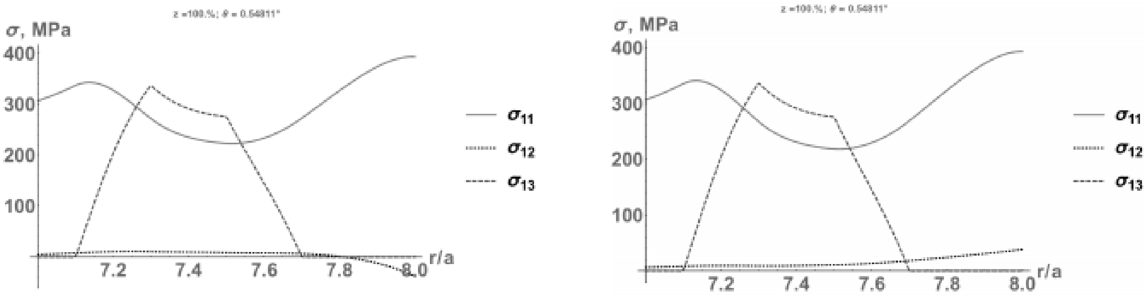
The Fig. 3 shows that maximal values of stress components at the outer rim of the disk are equal approximately 30-50 MPa, hence, the scope of stresses per torsion blade cycle is 60-100 MPa.

Earlier in [2] solved the problem about stress-strain state of the same disk for flight cycles under centrifugal forces in disks and periodic surface loading at the outer rim of the disc due to centrifugal and aerodynamics loads in blades.

For study of very high cycle fatigue (VHCF) due to vibrations it needs to impose the stress-strain state calculated here for the torsional vibrations of the blades with the signs

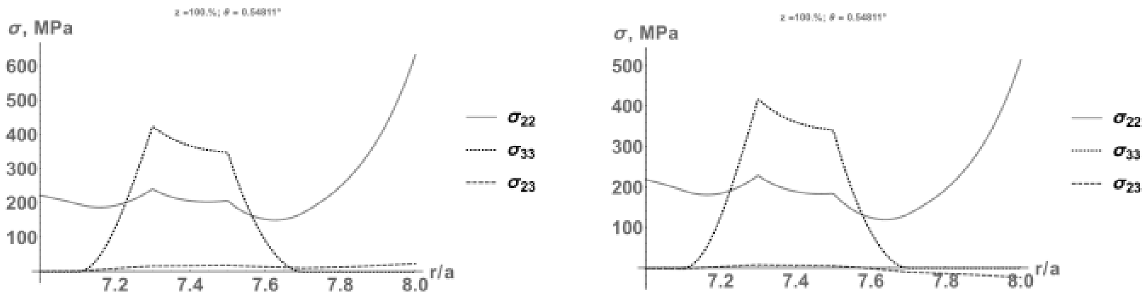
+ and - on the main stress-strain state associated with the flight loading cycles (LCF - low cycle fatigue).

The radial distributions of total stress components for the extreme positions in the cycle of torsional vibrations of the blades in the vicinity of the outer disk rim are shown below in Fig. 4-5 (a-b).



a) б)

Fig. 4. The total radial stress distribution.



a) б)

Fig. 5. The total radial stress distribution.

The difference between the values of the stresses on the left (a) and right (b) hand graphs in these figures represents the stress variation range in the high frequency cycle associated with the torsional vibrations of the blades. In future the data obtained by the proposed method are planned to be used in estimation of the durability of turbomachine disks.

Conclusions

A method is developed for calculating three-dimensional stress-strain state of elastic disks of variable thickness under the action of cyclic loads due to the torsional vibration of the blades in the gas turbine compressor.

Approximate representation of solutions is accepted for its dependence on the coordinates along the thickness of the disk and in the circumferential direction. For dependent on radial coordinate coefficients the system of ODE is derived and boundary value problems are formulated and solved by using an implicit finite difference scheme. Calculated stress-strain state due to blade vibrations superimposed on to the known stress-strain state of the disks due to centrifugal and aerodynamic loads. Total stress-strain state is ready for use in estimations of durability gas turbine disks.

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