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To cite this article: I S Nikitin *et al* 2021 *J. Phys.: Conf. Ser.* **1945** 012042

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# Mathematical Modeling of Fatigue Fracture at High-Frequency Bending Vibrations

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**Abstract.** A two-criterion kinetic damage model is proposed to describe the development of the fatigue fracture process under cyclic loading. On this basis, a procedure is proposed for calculating the coefficients of the kinetic equation for various modes of fatigue fracture from low-cycle to very-high-cycle fatigue. A uniform numerical method has been developed based on an implicit approximation of the differential equation for damage. This method makes it possible to carry out an end-to-end calculation of the evolution of crack-like zones of fatigue fracture in the material without considering cracks in their classical sense, as well as to estimate the durability of samples from the first nucleation to macrofracture. Calculations were made of the fatigue fracture of specimens under long-term cyclic loading according to the three-point bending scheme with the development of crack-like fracture zones in the modes from HCF to VHCF. Comparison of numerical and experimental results for specimens of titanium alloys is carried out.

## 1. Introduction

In this paper we studied the process of fatigue damage zones development. The damage theory was used to do so. As applied to cyclic loading and fatigue failure problems, this approach has been implemented in [1,2]. A complex model for the development of fatigue failure that uses the evolutionary equation for the damage function was proposed in [3,4].

Its basic (single-criterion) version was described in [3]. A two-criterion kinetic damage model is proposed to describe the development of the fatigue fracture process under cyclic loading [4]. To determine the coefficients of the kinetic equation of damage, the well-known criteria of multiaxial fatigue fracture were used: Smith-Watson-Topper (SWT) [5,6], which contains a mechanism associated with the development of normal opening microcracks, and Carpinteri-Spagnoli-Vantadori (CSV) [7], which contains a mechanism associated with the development of shear microcracks.

In a complex stress state in the proposed complex model, it is possible for any of the considered crack development mechanisms to be implemented. Cracks of different types may develop simultaneously in various parts of a specimen.

We used the following scheme for the amplitude fatigue curve. So-called repeated-static loading regime is realized up to  $N \sim 10^3$  value. In this regime the stress amplitude differs only slightly from the static tensile strength  $\sigma_{UTS}$ . Further, the fatigue curve (Wöhler curve or left branch of fatigue curve) describes the modes of the LCF-HCF up to  $N \sim 10^7$  with an asymptotic transition to the fatigue limit  $\sigma_u$ . Then, a zone of change in fracture mechanisms begins with a further drop in fatigue strength, starting from values of  $N \sim 10^8$ , to a new limiting value  $\tilde{\sigma}_u$  in accordance with the right branch of the bimodal fatigue  $S-N$  curve (VHCF mode) [8].



We managed to determine the model parameters for various fatigue failure modes in a range that spans from low-cycle fatigue (LCF) and high-cycle fatigue (HCF) to very-high-cycle fatigue (VHCF) [9]. Also a numerical method for calculating crack-like zones up to macro fracture based on the proposed complex model was proposed [3,4]. Here the proposed damage model and numerical procedure are used to reproduce results of fatigue VHCF three-point bending tests [10,11] and the corresponding experimental fatigue curves. The operability of the model and calculation algorithm are verified.

## 2. Kinetic equation for damage in LCF-HCF mode

Different criteria use different combinations of stresses to calculate an equivalent stress value. Some of them are based on normal components of the stress state, while others use shear stress components. We wanted to implement two criteria simultaneously. The first one is the stress-based Smith–Watson–Topper [5,6] criterion and implies normal-opening micro-cracks. The second one is the stress-based Carpinteri–Spagnoli–Vantadori [7] criterion and implies the notion of a critical plane and shear micro-cracks. The considered model develops the damage model in case of cyclic loads, presented in [12] for the description of damages during dynamic loading.

The generic fatigue fraction criterion corresponding to the left branch of the bimodal fatigue curve in the following has the form:

$$\sigma_{eq} = \sigma_u + \sigma_L N^{-\beta_L} \quad (1)$$

In the repeated-static fracture region that is up to  $N \sim 10^3$  value it is possible to obtain the  $\sigma_L = 10^{3\beta} (\sigma_{UTS} - \sigma_u)$  values by the method [9]. Here  $\sigma_{UTS}$  is the static tensile strength of the material,  $\sigma_u$  is the classic fatigue limit of the material during a reverse cycle (asymmetry coefficient of the cycle  $R = -1$ ),  $\beta_L$  is power index of the left branch of the bimodal fatigue curve.

In order to describe the process of fatigue damage development in the LCF-HCF mode, a damage function  $0 \leq \psi(N) \leq 1$  is introduced, which describes the process of gradual cyclic material failure. When  $\psi = 1$  a material particle is considered completely destroyed. Its Lamé modules become equal to zero. The damage function  $\psi$  as a function on the number of loading cycles for the LCF-HCF mode is described by the kinetic equation:

$$\partial \psi / \partial N = B_L \psi^\gamma / (1 - \psi^\alpha) \quad (2)$$

where  $\alpha$  and  $0 < \gamma < 1$  are the model parameters that determine the rate of fatigue damage development. The choice of the denominator in this two-parameter equation, which sets the infinitely large growth rate of the zone of complete failure at  $\psi \rightarrow 1$ , is determined by the known experimental data on the kinetic growth curves of fatigue cracks, which have a vertical asymptote and reflect the fact of their explosive, uncontrolled growth at the last stage of macro fracture.

An equation for damage of a similar type was considered in [2], its numerous parameters and coefficients were determined indirectly from the results of uniaxial fatigue tests. In our case, the coefficient  $B_L$  is determined by the procedure that is clearly associated with the selected criterion for multiaxial fatigue failure of one type or another. The expression for the coefficient  $B_L$  has a form [4]:

$$B_L = 10^{-3} \left[ \langle \sigma_{eq} - \sigma_u \rangle / (\sigma_{UTS} - \sigma_u) \right]^{1/\beta_L} \alpha / (1 + \alpha - \gamma) / (1 - \gamma) \quad (3)$$

where the  $\sigma_{eq}$  value is determined by the selected fatigue fracture mechanism and the corresponding multiaxial criterion.

### 2.1. SWT criterion

The criterion of multiaxial fatigue failure in the LCF-HCF mode with the development of normal-stress micro-cracks (stress-based SWT) corresponding to the left branch of the bimodal fatigue curve has the form:

$$\sqrt{\langle \sigma_{1_{\max}} \rangle \Delta \sigma_1 / 2} = \sigma_u + \sigma_L N^{-\beta_L} \quad (4)$$

where  $\sigma_1$  is the largest principal stress,  $\Delta \sigma_1$  is the spread of the largest principal stress per cycle,  $\Delta \sigma_1 / 2$  is its amplitude. According to the chosen criterion, only tensile stresses lead to failure, so it has the value  $\langle \sigma_{1_{\max}} \rangle = \sigma_{1_{\max}} H(\sigma_{1_{\max}})$ .

Let us introduce the following notation:

$$\sigma^n = \sqrt{\langle \sigma_{1_{\max}} \rangle \Delta \sigma_1 / 2} \quad (5)$$

Here the upper index  $n$  stands for denotation and should not be considered as a power.

### 2.2. CSV criterion

The criterion of multiaxial fatigue failure in the LCF-HCF mode, including the concept of a critical plane (stress-based CSV), corresponding to the left branch of the bimodal fatigue curve has the form:

$$\sqrt{(\langle \Delta \sigma_n \rangle / 2)^2 + k_c^2 (\Delta \tau_n / 2)^2} = \sigma_u + \sigma_L N^{-\beta_L} \quad (6)$$

where  $\Delta \tau_n / 2$  is the amplitude of the tangential stress on the plane (critical), where it reaches its maximum value,  $\Delta \sigma_n / 2$  is the amplitude of the normal (tensile) stress on the critical plane,  $\langle \Delta \sigma_n \rangle = \Delta \sigma_n H(\sigma_{n_{\max}})$ . Here, the shear fatigue limit  $\tau_u$  for a pulsating cycle is additionally introduced at a cycle asymmetry coefficient of  $R = -1$ . In a simplified formulation, we can assume  $k_c \approx \sigma_u / \tau_u$  and  $k_c \approx \sqrt{3}$ . This criterion includes the mechanism of fatigue fracture with the formation of shear micro-cracks.

Let us introduce the following notation:

$$\sigma^\tau = \sqrt{(\langle \Delta \sigma_n \rangle / 2)^2 + 3(\Delta \tau_n / 2)^2} \quad (6)$$

Here the upper index  $\tau$  stands for denotation and should not be considered as a power.

### 2.3. Criterion selection

At each node there are not one but two  $B_L$  values, namely  $B_L^n$  and  $B_L^\tau$ . They have the forms:

$$B_L^n = 10^{-3} \left[ \langle \sigma^n - \sigma_u \rangle / (\sigma_{UTS} - \sigma_u) \right]^{1/\beta_L} \alpha / (1 + \alpha - \gamma) / (1 - \gamma) \quad (7)$$

$$B_L^\tau = 10^{-3} \left[ \langle \sigma^\tau - \sigma_u \rangle / (\sigma_{UTS} - \sigma_u) \right]^{1/\beta_L} \alpha / (1 + \alpha - \gamma) / (1 - \gamma) \quad (8)$$

Presence of two  $B_L$  values means that there are two independent damage values. They are  $\psi^n = f(B_L^n)$  and  $\psi^\tau = f(B_L^\tau)$ . At each iteration of a numerical calculation process, both  $\psi^n$  and  $\psi^\tau$  are calculated for every node. Then the greatest of two is selected to be used in element's material properties modification. The resulting formulas for the coefficients of the kinetic equation for damage operate in the range  $\sigma_u < \sigma_{eq} \leq \sigma_{UTS}$ .

### 3. Kinetic equation for damage in VHCF mode

The criterion for multiaxial fatigue failure in the VHCF mode corresponding to the right branch of the bimodal fatigue curve has the form:

$$\sigma_{eq} = \tilde{\sigma}_u + \sigma_V N^{-\beta_V} \quad (9)$$

We will assume that the choice of  $\sigma_{eq}$  is determined by the same mechanisms of microcracks development and fatigue fracture criteria SWT and CSV as in the HCF mode. For microcracks of normal opening  $\sigma_{eq} = \sigma^n$ , for shear  $\sigma_{eq} = \sigma^\tau$ .

From the condition of similarity of the reference points for the left and right branches of the bimodal fatigue curve [8], one can obtain the formula  $\sigma_V = 10^{8\beta_V} (\sigma_u - \tilde{\sigma}_u)$ . Here  $\tilde{\sigma}_u$  is the fatigue limit of the material in the reverse cycle for the VHCF mode,  $\beta_V$  is the power exponent of the right branch of the bimodal fatigue curve.

For the VHCF mode, it is possible to determine the coefficient in the evolutionary equation for damage:

$$d\psi/dN = B_V \psi^\gamma / (1 - \psi^\alpha), \quad 0 < \alpha, \gamma < 1 \quad (10)$$

As in the previous section, we can obtain expressions for the coefficients of the kinetic equation of damage in the VHCF mode:

$$B_V^n = 10^{-8} \left[ \langle \sigma^n - \tilde{\sigma}_u \rangle / (\sigma_u - \tilde{\sigma}_u) \right]^{1/\beta_V} \alpha / (1 + \alpha - \gamma) / (1 - \gamma) \quad (11)$$

$$B_V^\tau = 10^{-8} \left[ \langle \sigma^\tau - \tilde{\sigma}_u \rangle / (\sigma_u - \tilde{\sigma}_u) \right]^{1/\beta_V} \alpha / (1 + \alpha - \gamma) / (1 - \gamma) \quad (12)$$

The resulting formulas for the coefficients of the kinetic equation for damage operate in the range  $\tilde{\sigma}_u < \sigma_{eq} \leq \sigma_u$ .

#### 4. Calculation algorithm

The Ansys software was used to calculate the loading cycle of a deformable specimen, supplemented by a code to calculate the damage equation and changes of elasticity modulus.

A uniform numerical method has been developed based on implicit integration of the differential equation for damage [3,4]. To integrate the damage equation  $d\psi/dN = B\psi^\gamma / (1 - \psi^\alpha)$  where  $B = B_L, B_V$ , the approximation for the damage function was applied at a  $k$ -node of the mesh. This function takes given discrete  $\psi_k^t$  value at  $N^t$  moment and returns sought  $\psi_k^{t+1}$  value at  $N^{t+1}$  moment.

We decided to simplify the damage equation by making the  $\alpha = 1 - \gamma$  substitution. We were driven by the fact that the damage function parameters describe the material properties, so there is no need to put different variables to describe the same thing. After that we analytically integrated the damage equation to obtain an explicit expression for  $\psi_k^{t+1}(\psi_k^t, \Delta N^t)$  [3]:

$$\psi_k^{t+1} = \left( 1 - \sqrt{1 - [2(1 - \gamma)B\Delta N^t + 2(\psi_k^t)^{1-\gamma} - (\psi_k^t)^{2(1-\gamma)}]} \right)^{1/(1-\gamma)} \quad (13)$$

At each node the  $B$  coefficient is calculated. It is based on the stress at the corresponding node. Next, the values from (14) are calculated for every single node. This values is the number of cycles after which node  $k$  with corresponding stress level and damage value is going to reach the ultimate damage level when the damage is equal to 1.

$$\Delta \tilde{N}_k^t = \left[ \psi^{1-\gamma} / (1 - \gamma) - \psi^{2(1-\gamma)} / 2 / (1 - \gamma) \right]_{\psi_k^t}^1 / B \quad (14)$$

If the damage level in the node  $k$  is less than the threshold level  $\psi_0$  (threshold  $\psi_0 = 0.95$  is selected), then the value for this node  $\Delta \tilde{N}_k^t$  is multiplied by a factor of 0.5. Otherwise, it is multiplied by a factor of 1. Thus, the step of incrementing the number of cycles for a given node is  $\Delta N_k^t = 0.5(1 + H(\psi_k^t - 0.95))\Delta \tilde{N}_k^t$ . Of all the  $\Delta N_k^t$  values the smallest one is selected. The increment of the number of loading cycles for the calculation of the entire specimen is  $\Delta N^t = \min_k \Delta N_k^t$ . For each

node, based on its current level of damage and equivalent stress, a new level of damage is found taking into account the calculated increment  $\Delta N^t$ .

#### 4.1. Material properties change

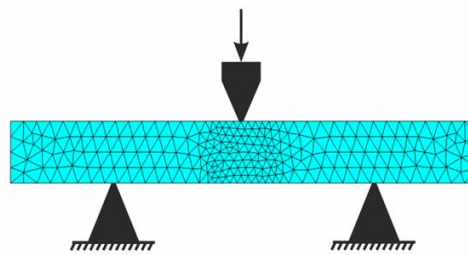
All elements are sorted out, for each of them the most damaged node is searched and according to its damage the mechanical properties of the element are adjusted:

$$\lambda(\psi_k^t) = \lambda_0(1 - \kappa\psi_k^t), \quad \mu(\psi_k^t) = \mu_0(1 - \kappa\psi_k^t) \quad (15)$$

The elements that are comprise at least on node with the ultimate level of damage  $\psi = 1$  are excluded from the specimen and form a localized (crack-like) zone of completely destroyed material. Depending on our goal we either stop the calculation after a crack initiation happen or after a crack grows to a certain desired length (macro destruction). Sometimes a crack growth process stops and is the third way to end the calculation.

### 5. Calculation results

We conducted the series of numerical experiments in order to determine the impact of our proposed scheme on fatigue behavior of a specimen. Calculations were made of the fatigue fracture of specimens under cyclic loading according to the three-point bending scheme with the development of crack-like fracture zones in the modes from HCF to VHCF. An analytical calculation of the geometric

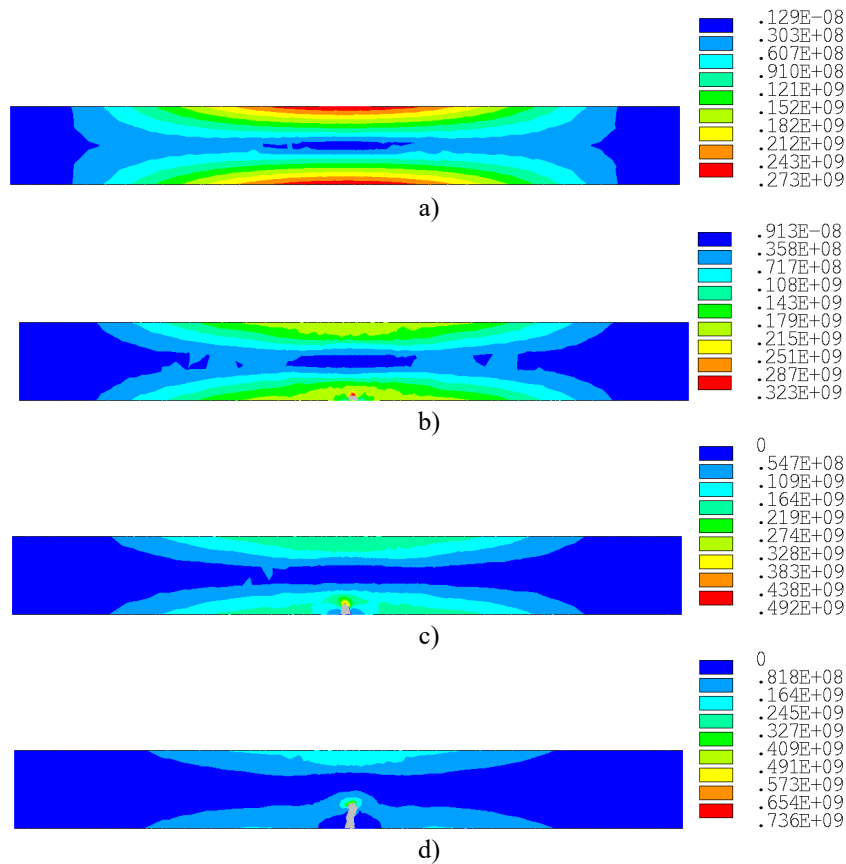


**Figure 1.** FEM mesh, boundary conditions, and implemented loading

and frequency parameters of this loading scheme in the elastic mode is given in [10].

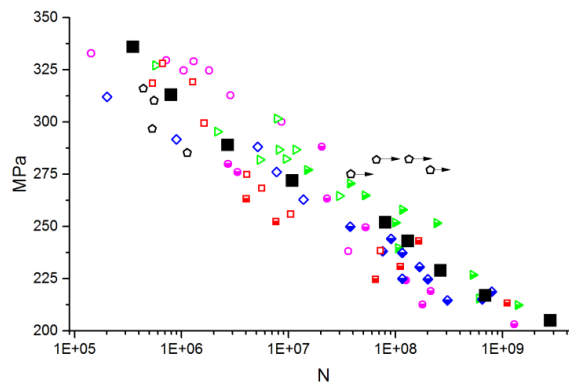
Comparison of numerical and experimental results for specimens from titanium alloys Ti-6Al-4V is carried out. We used the experimental results obtained by the method [11]. The physical properties are as follows:  $\sigma_{UTS} = 900$  MPa,  $\sigma_u = 245$  MPa,  $\tilde{\sigma}_u = 185$  MPa,  $\rho = 4500$  kg/m<sup>3</sup>,  $\nu = 0.37$ ,  $\beta_L = 0.38$ ,  $\beta_V = 0.33$ .

The size of the bar is 26 x 6.5 x 3 mm. The computational mesh with the high-frequency loading scheme is shown in figure 1. The distance between the mesh nodes in the center is 0.4 mm and at the edges is 0.8 mm.



**Figure 2.** The field of effective stresses at the initial moment - a), nucleation of a quasi-crack, 3.24E9 cycles - b), quasi-crack development, 3.25E9 cycles - c), breaking of quasi-crack, 3.25E9 cycles - d)

The effective stress fields at different values of the number of cycles, as well as the development of zones of fatigue damage ("quasi-cracks") are shown in figure 2 in gray color. The results of numerical simulation of fatigue curves obtained in three-point bending tests [11] are shown in figure 3.



**Figure 3.** Calculated fatigue curve (black rectangles) and experimental data [11].

The experimental results obtained for a titanium alloy with various processing technologies are marked by colours in figure 3. The influence of these processing technologies is not taken into account in the proposed model. Nevertheless, the calculated data correlate well with the experimental data. We also note that the considered range of cyclic tests covers range from  $N \sim 10^5$  to  $N \sim 10^9$  (HCF and VHCF

modes, respectively), but the numerical simulation of fatigue experiments for three-point bending was carried out within the framework of a complex damage model using a uniform numerical procedure.

## 6. Conclusions

A two-criterion kinetic damage model is proposed to describe the development of the fatigue fracture process under cyclic loading. On this basis, a procedure is proposed for calculating the coefficients of the kinetic equation for various modes of fatigue fracture from low-cycle to very-high-cycle fatigue.

An implicit scheme is used for numerical decision of the damage equation and calculation of the crack-like zones development. The durability of the specimens from the fatigue fracture nucleation to macrofracture is estimated.

Calculations were made of the fatigue fracture of specimens under long-term cyclic loading according to the three-point bending scheme with the development of crack-like fracture zones in the modes from HCF to VHCF. Comparison of numerical and experimental results for specimens of titanium alloys is carried out.

## Acknowledgments

This study is supported by Russian Science Foundation (grant № 19-19-00705).

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