SURFACE WAVES OF NON-RAYLEIGH TYPE

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Abstract. Existence of surface waves of non-Rayleigh type propagating on some anisotropic elastic half-spaces is proved. Conditions for originating the non-Rayleigh type waves are analyzed. An example of a transversely isotropic material admitting a surface wave of the non-Rayleigh type, is constructed.

1. Introduction. In our previous paper [1] it was shown that some anisotropic elastic materials exhibit property of non existence of the genuine Rayleigh waves:

$$\mathbf{u}(\mathbf{x}) = \sum_{k=1}^{3} C_k \mathbf{m}_k e^{ir(\gamma_k \, \boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} \tag{1.1}$$

where C_k are complex coefficients determined up to a multiplier by the traction-free boundary conditions; $\mathbf{m}_k$ are complex eigenvectors of the Christoffel equation, which will be introduced further; these eigenvectors correspond to complex roots γ_k of the characteristic polynomial; r is the (real) wave number; ν is an outward normal to the boundary Π_ν of the half-space along which the surface wave propagates; $\mathbf{n} \in \Pi_\nu$ is the unit vector determining direction of propagation of the surface wave, and c is the phase speed. The terms

$$\mathbf{u}_k(\mathbf{x}) \equiv \mathbf{m}_k e^{ir(\gamma_k \boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} \tag{1.2}$$

are called *partial waves*.

As was shown in [1], the existence of the “forbidden” directions or “forbidden” planes along which the genuine Rayleigh wave cannot propagate is due to appearing the Jordan blocks in a specially constructed 6×6-matrix associated with the Christoffel equation. The following analysis reveals that the situation regarded in [1] appears to be more complicated. The Jordan blocks in the regarded matrix lead to a qualitative change of the structure of the partial waves (1.2) and, while the genuine Rayleigh wave at the situation considered in [1] does not exist, there remains an exponentially attenuating with depth surface wave of the non-Rayleigh type.

2. Basic notations. Equations of motion for anisotropic elastic medium can be written in the form

$$\mathbf{A}(\partial_x, \ \partial_t)\mathbf{u} \equiv \mathrm{div}_x \, \mathbf{C} \cdot\cdot \nabla_x \mathbf{u} - \rho\ddot{\mathbf{u}} = 0 \tag{2.1}$$

where $\mathbf{u}$ is the displacement field; ρ is the density of a medium; and $\mathbf{C}$ is the fourth-order elasticity tensor assumed to be *positive definite*:

$$\underset{\mathbf{A}\in sym(R^3\otimes R^3), \ \mathbf{A}\neq 0}{\forall \mathbf{A}} \qquad (\mathbf{A}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{A}) \equiv \sum_{i,j,\ m,n} A_{ij}C^{ijmn}A_{mn} > 0 \tag{2.2}$$

The sign "$\cdot$" in (2.1), (2.2) and henceforth means the scalar multiplication in the corresponding unitary or Euclidian vector space.

Substituting partial waves (1.2) in Eq. (2.1) produces the Christoffel equation:

$$\left[(\gamma_k \boldsymbol{\nu} + \mathbf{n}) \cdot \mathbf{C} \cdot (\mathbf{n} + \gamma_k \boldsymbol{\nu}) - \rho c^2 \mathbf{I} \right] \cdot \mathbf{m}_k = 0 \tag{2.3}$$

where $\mathbf{I}$ is the unit diagonal matrix. Equation (2.3) can be written in the equivalent form:

$$\det\left[(\gamma_k \boldsymbol{\nu} + \mathbf{n}) \cdot \mathbf{C} \cdot (\mathbf{n} + \gamma_k \boldsymbol{\nu}) - \rho c^2 \mathbf{I} \right] = 0 \tag{2.4}$$

The left-hand side of Eq. (2.4) represents a polynomial of degree 6 with respect to γ_k.

REMARK 2.1. It can be shown, see [1], that if the phase speed does not exceed the so called lower limiting speed ($c_3^{\lim}$):

$$c < c_3^{\lim}, \tag{2.5}$$

then all the roots of Eq. (2.3) are complex with $\mathrm{Im}(\gamma_k) \neq 0$. The inequality (2.5) ensures that three partial waves (1.2) with $\mathrm{Im}(\gamma_k) < 0$ attenuate with depth in a "lower" half-space at $(\boldsymbol{\nu} \cdot \mathbf{x}) < 0$. Only attenuating with depth partial waves, as being physically reasonable, will be considered further.

3. Six-dimensional formalism. Following [1], a more general representation for the partial wave than (1.2), will be considered:

$$\mathbf{v}(x'')\, e^{ir(\mathbf{n}\cdot\mathbf{x} - ct)} \tag{3.1}$$

where $x'' = ir\,\boldsymbol{\nu}\cdot\mathbf{x}$ is the dimensionless complex coordinate, $\mathbf{v}(x'')$ is an unknown vector function, and the exponential multiplier in (3.1) corresponds to propagation of the plane wave front along the direction $\mathbf{n}$ with the phase speed c. Substituting representation (3.1) into Eq. (2.1) yields the following system of ordinary differential equations:

$$\left((\nu\cdot\mathbf{C}\cdot\nu)\partial^2_{x''}+(\nu\cdot\mathbf{C}\cdot\mathbf{n}+\mathbf{n}\cdot\mathbf{C}\cdot\nu)\partial_{x''}-(\mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n}-\rho c^2\mathbf{I}\ )\right)\mathbf{v}(x'')=0 \tag{3.2}$$

Direct analysis of system (3.2) is rather difficult, and reduction to the first-order system can simplify it.

Introduction of a new vector-function $\mathbf{w}=\partial_{x''}\mathbf{v}$ allows us to reduce the second-order system (3.2) in C^3 to the first-order one in C^6:

$$\partial_{x''}\begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix} = \mathbf{R}_6\cdot\begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix} \tag{3.3}$$

In (3.3) the complex six-dimensional matrix $\mathbf{R}_6$ has the form

$$\mathbf{R}_6=\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M} & -\mathbf{N}\end{pmatrix} \tag{3.4}$$

where three-dimensional matrices $\mathbf{M}$ and $\mathbf{N}$ have the form

$$\begin{aligned}\mathbf{M}&=(\nu\cdot\mathbf{C}\cdot\nu)^{-1}\cdot(\mathbf{n}\cdot\mathbf{C}\ \cdot\mathbf{n}-\rho c^2\mathbf{I}\ )\\ \mathbf{N}&=(\nu\cdot\mathbf{C}\cdot\nu)^{-1}\cdot(\nu\cdot\mathbf{C}\cdot\mathbf{n}+\mathbf{n}\cdot\mathbf{C}\cdot\nu)\end{aligned} \tag{3.5}$$

In (3.4) $\mathbf{I}$ stands for the unit (diagonal) matrix in the three-dimensional space.

A surjective homomorphism $\Im$: $C^6\to C^3$, such that

$$\Im(\mathbf{v},\mathbf{w})=\mathbf{v} \tag{3.6}$$

will be needed for the subsequent analysis.

The following Proposition takes place [1]:

PROPOSITION 3.1. Let $c\in(0;\ \ c_3^{\lim})$:

a) Spectrum of the matrix $\mathbf{R}_6$ coincides with the set of all roots of polynomial (2.4);

b) If γ is a complex eigenvalue and $\mathbf{m}=(\mathbf{m}',\ \mathbf{m}'')$, $\mathbf{m}',\ \mathbf{m}''\in C^3$ is the corresponding six-dimensional eigenvector of the matrix $\mathbf{R}_6$, then $\bar\gamma$ is also an eigenvalue with the corresponding eigenvector $\overline{\mathbf{m}}=(\overline{\mathbf{m}'},\ \overline{\mathbf{m}''})$;

c) The matrix $\mathbf{R}_6$ admits the following Jordan normal forms

$$\mathbf{J}_6^{(\mathrm{I})} = \begin{pmatrix} \gamma_1 &&&&& \\ & \bar{\gamma}_1 &&&& \\ && \gamma_2 &&& \\ &&& \bar{\gamma}_2 && \\ &&&& \gamma_3 & \\ &&&&& \bar{\gamma}_3 \end{pmatrix}, \quad \mathbf{J}_6^{(\mathrm{II})} = \begin{pmatrix} \begin{pmatrix} \gamma_1 & 1 \\ 0 & \gamma_1 \end{pmatrix} &&& \\ & \begin{pmatrix} \bar{\gamma}_1 & 1 \\ 0 & \bar{\gamma}_1 \end{pmatrix} && \\ && \gamma_3 & \\ &&& \bar{\gamma}_3 \end{pmatrix},$$

$$\mathbf{J}_6^{(\mathrm{III})} = \begin{pmatrix} \begin{pmatrix} \gamma_1 & 1 & 0 \\ 0 & \gamma_1 & 1 \\ 0 & 0 & \gamma_1 \end{pmatrix} & \\ & \begin{pmatrix} \bar{\gamma}_1 & 1 & 0 \\ 0 & \bar{\gamma}_1 & 1 \\ 0 & 0 & \bar{\gamma}_1 \end{pmatrix} \end{pmatrix} \tag{3.7}$$

d) According to the Jordan normal forms the following three types of representations for surface waves occur:

(i) for the Jordan normal form $\mathbf{J}_6^{(\mathrm{I})}$, the corresponding representation is given by (1.1);

(ii) for the Jordan normal form $\mathbf{J}_6^{(\mathrm{II})}$, the representation is as follows:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) = (C_1 + ir\,C_2\,\nu\cdot\mathbf{x})\,\mathbf{m}_1'\,e^{ir(\gamma_1\,\nu\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} &+ \\ C_2\,\mathbf{m}_2'\,e^{ir(\gamma_1\,\nu\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} &+ \\ C_3\,\mathbf{m}_3'\,e^{ir(\gamma_3\,\nu\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} & \end{aligned} \tag{3.8}$$

where $\mathbf{m}_1' = \Im(\mathbf{m}_1) \in C^3$, and $\mathbf{m}_1$ is the eigenvector of $\mathbf{R}_6$ corresponding to the eigenvalue γ_1; $\mathbf{m}_2' = \Im(\mathbf{m}_2) \in C^3$, and $\mathbf{m}_2 \in C^3$ is the generalized eigenvector associated with $\mathbf{m}_1$, and the eigenvector $\mathbf{m}_3 \in C^6$ corresponds to the eigenvalue γ_3;

(iii) for the Jordan normal form $\mathbf{J}_6^{(\mathrm{III})}$, the representation is as follows:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) = (C_1 + ir\,C_2\nu\cdot\mathbf{x} + \tfrac{1}{2}C_3\,(ir\,\nu\cdot\mathbf{x})^2)\,\mathbf{m}_1'\,e^{ir(\gamma_1\,\nu\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} &+ \\ (C_2 + ir\,C_3\nu\cdot\mathbf{x})\,\mathbf{m}_2'\,e^{ir(\gamma_1\,\nu\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} &+ \\ C_3\,\mathbf{m}_3'\,e^{ir(\gamma_1\,\nu\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} & \end{aligned} \tag{3.9}$$

$\mathbf{m}_1' = \Im(\mathbf{m}_1) \in C^3$, and $\mathbf{m}_1$ is the eigenvector corresponding to the eigenvalue γ_1; and $\mathbf{m}_2, \mathbf{m}_3 \in C^6$ are the generalized eigenvectors associated with $\mathbf{m}_1$.

COROLLARY. For any of the Jordan normal forms of the matrix $\mathbf{R}_6$ the three-dimensional components $\mathbf{m}_k'$, $\mathbf{m}_k''$ of the (proper) eigenvector $\mathbf{m}_k$, satisfy the equations

$$\begin{aligned} &\mathbf{m}_k'' = \gamma_k\, \mathbf{m}_k' \\ &\left(\gamma_k{}^2\mathbf{I} + \gamma_k\mathbf{N} + \mathbf{M}\right)\cdot \mathbf{m}_k' = 0 \end{aligned} \tag{3.10}$$

Proof. When the matrix $\mathbf{R}_6$ has no Jordan blocks, the solution of Eq. (3.3) in view of (3.4) leads to Eqs. (3.10). Thus, the component $\mathbf{m}_k'$ belongs to the kernel space of the matrix $\left(\gamma_k{}^2\mathbf{I} + \gamma_k\mathbf{N} + \mathbf{M}\right)$.

4. Construction of the generalized eigenvector for $\mathbf{J}_6^{(\mathrm{II})}$. In view of [2] the solution of Eq. (3.3) corresponding to a Jordan block of the second rank can be represented in the form

$$\left(C_1\left(\mathbf{m}_1',\ \mathbf{m}_1''\right) + C_2\left(x''\left(\mathbf{m}_1',\ \mathbf{m}_1''\right) + \left(\mathbf{m}_2',\ \mathbf{m}_2''\right)\right)\right)e^{\gamma_1 x''} \tag{4.1}$$

where as before $x'' = ir\,\nu\cdot\mathbf{x}$.

PROPOSITION 4.1. a) The three-dimensional components $\mathbf{m}_1'$, $\mathbf{m}_1''$ of the genuine eigenvector satisfy Eqs. (3.10);

b) Components $\mathbf{m}_2'$, $\mathbf{m}_2''$ of the generalized eigenvector satisfy the following equations:

$$\begin{aligned} &\left(\gamma_1{}^2\mathbf{I} + \gamma_1\mathbf{N} + \mathbf{M}\right)\cdot \mathbf{m}_2' = -\left(2\gamma_1\mathbf{I} + \mathbf{N}\right)\cdot \mathbf{m}_1' \\ &\mathbf{m}_2'' = \mathbf{m}_1' + \gamma_1\mathbf{m}_2' \end{aligned} \tag{4.2}$$

c) At $c \in (0;\ \ c_3^{\lim})$ the matrix $\left(2\gamma_1\mathbf{I} + \mathbf{N}\right)$ is not degenerate;

d) At $c \in (0;\ \ c_3^{\lim})$ vectors $\left(2\gamma_1\mathbf{I} + \mathbf{N}\right)\cdot\mathbf{m}_1'$ and $\mathbf{m}_1'\cdot(\nu\cdot\mathbf{C}\cdot\nu)$ are orthogonal.

Proof. Conditions a) and b) flow out by direct substituting the solution (4.1) into Eq. (3.3).

To prove c) it is sufficient to demonstrate that the matrix

$$(\nu\cdot\mathbf{C}\cdot\nu)\cdot\left(2\gamma_1\mathbf{I} + \mathbf{N}\right) = 2\gamma_1(\nu\cdot\mathbf{C}\cdot\nu) + (\nu\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\nu) \tag{4.3}$$

is not degenerate. Considering multiplication of the right-hand side of (4.3) by any nonzero conjugate complex vectors $\mathbf{a}, \bar{\mathbf{a}} \in C^3$ and accounting Remark 2.1, which ensures $\mathrm{Im}(\gamma_1) \neq 0$, we arrive to

$$\begin{aligned} &\mathrm{Im}\left(\mathbf{a} \cdot \left(2\gamma_1(\nu \cdot \mathbf{C} \cdot \nu) + (\nu \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \nu)\right) \cdot \bar{\mathbf{a}}\right) = \\ &\qquad 2\,\mathrm{Im}(\overline{\gamma_1})(\mathbf{a} \otimes \nu \cdot\cdot\, \mathbf{C} \cdot\cdot\, \nu \otimes \bar{\mathbf{a}}) \neq 0 \end{aligned} \tag{4.4}$$

In obtaining (4.4) we took into consideration that $\mathrm{Im}\left(\mathbf{a} \cdot \left(\nu \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \nu\right) \cdot \bar{\mathbf{a}}\right) = 0$, since the matrix $\left(\nu \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \nu\right)$ is (real) symmetric. The last inequality in (4.4) completes the proof of condition c).

To prove d) Eq. (4.2) can be transformed into equivalent one by multiplying both sides by the nondegenerate matrix $(\nu \cdot \mathbf{C} \cdot \nu)$, this gives

$$\begin{aligned} &\left(\gamma_1^2(\nu \cdot \mathbf{C} \cdot \nu) + \gamma_1(\nu \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \nu) + (\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} - \rho c^2 \mathbf{I})\right) \cdot \mathbf{m}_2' = \\ &\qquad -\left(2\gamma_1(\nu \cdot \mathbf{C} \cdot \nu) + (\nu \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \nu)\right) \cdot \mathbf{m}_1' \end{aligned} \tag{4.5}$$

Now, the vector $\mathbf{m}_1'$ belongs to the kernel space of the matrix in the left-hand side of Eq. (4.5), which flows out from Proposition 4.1.a. Moreover, the regarded matrix is complex symmetric, hence its left and right eigenvectors coincide. The latter allows us to write for the left-hand side of Eq. (4.5)

$$\mathbf{m}_1' \cdot \left(\gamma_1^2(\nu \cdot \mathbf{C} \cdot \nu) + \gamma_1(\nu \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \nu) + (\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} - \rho c^2 \mathbf{I})\right) \cdot \mathbf{m}_2' = 0 \tag{4.6}$$

Similarly, for the right-hand side of Eq. (4.5)

$$\mathbf{m}_1' \cdot \left(2\gamma_1(\nu \cdot \mathbf{C} \cdot \nu) + (\nu \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \nu)\right) \cdot \mathbf{m}_1' = 0 \tag{4.7}$$

In view of (3.5), Eq. (4.7) completes the proof.

COROLLARY. In the factor-space $C^3 / \mathrm{Ker}\left(\gamma_1^2 \mathbf{I} + \gamma_1 \mathbf{N} + \mathbf{M}\right)$, the vector $\mathbf{m}_2'$ admits the following representation

$$\mathbf{m}_2' = -\left(\gamma_1^2 \mathbf{I} + \gamma_1 \mathbf{N} + \mathbf{M}\right)^{-1} \cdot \left(2\gamma_1 \mathbf{I} + \mathbf{N}\right) \cdot \mathbf{m}_1' \tag{4.8}$$

REMARK 4.1. a) At the regarded speed interval $c \in (0;\ c_3^{\lim})$ the eigenvectors of the complex symmetric matrix appearing in Eq. (4.6) may not form a set of mutually orthogonal vectors in C^3, in contrast to the mutually orthogonal eigenvectors of any real symmetric matrix.

b) For supersonic Lamb waves propagating with the phase speed exceeding the greatest limiting speed $c_1^{\lim}$, all eigenvalues of the matrix $\mathbf{R}_6$ become real. Presumably, in such a case the condition c) of Proposition 4.1 and the subsequent Corollary can be violated.

5. Dispersion equation for $\mathbf{J}_6^{(\mathrm{II})}$. The traction-free boundary conditions on the surface Π_ν can be written in the form:

$$\mathbf{t}_\nu \equiv \nu \cdot \mathbf{C} \cdot\cdot \nabla \mathbf{u}\Big|_{\mathbf{x}\in\Pi_\nu} = 0 \tag{5.1}$$

Substituting the displacement field into Eq. (5.1) yields

$$\sum_{k=1}^{3} C_k \mathbf{t}_k = 0 \tag{5.2}$$

where $\mathbf{t}_k$ are the partial surface traction.

The following two cases for the partial surface traction fields will be considered:

(i) For the Jordan normal form $\mathbf{J}_6^{(\mathrm{I})}$ and the representation (1.1), the partial surface tractions are of the form

$$\mathbf{t}_k = \left(\gamma_k\, \nu \cdot \mathbf{C} \cdot \nu + \nu \cdot \mathbf{C} \cdot \mathbf{n}\right) \cdot \mathbf{m}_k'\, e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \tag{5.3}$$

(ii) For the Jordan normal form $\mathbf{J}_6^{(\mathrm{II})}$ and the representation (3.8), the partial surface tractions are of the form

$$\begin{aligned}
\mathbf{t}_1 &= \left(\gamma_1\, \nu \cdot \mathbf{C} \cdot \nu + \nu \cdot \mathbf{C} \cdot \mathbf{n}\right) \cdot \mathbf{m}_1'\, e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \\
\mathbf{t}_2 &= \left(\gamma_1 \left(\nu \cdot \mathbf{C} \cdot \nu\right) \cdot \mathbf{m}_1' + \left(\gamma_1\, \nu \cdot \mathbf{C} \cdot \nu + \nu \cdot \mathbf{C} \cdot \mathbf{n}\right) \cdot \mathbf{m}_2' \right) e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \\
\mathbf{t}_3 &= \left(\gamma_3\, \nu \cdot \mathbf{C} \cdot \nu + \nu \cdot \mathbf{C} \cdot \mathbf{n}\right) \cdot \mathbf{m}_3'\, e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)}
\end{aligned} \tag{5.4}$$

Equations (5.2) can be regarded as linear system with respect to the unknown coefficients C_k. The existence of the nontrivial solution of Eqs. (5.2) is equivalent to vanishing of the following determinant:

$$\mathbf{t}_1 \wedge \mathbf{t}_2 \wedge \mathbf{t}_3 = 0 \tag{5.5}$$

Equation (5.5) provides a necessary and sufficient condition for the existence of the surface wave.

Equation (5.5) is known as the dispersion equation despite the fact, that the phase speed determined by this equation does not depend upon the wave number, or the wave frequency.

6. Surface waves of non-Rayleigh type in transversely isotropic media. Let the unit vectors $\mathbf{e}_k$, $k=1,2,3$ form an orthogonal basis in R^3, and vector $\mathbf{e}_1$ is normal to the Π_ν-basal plane of a transversely isotropic medium. This ensures that vector $\mathbf{e}_1$ and ν coincide. The corresponding elasticity tensor has the following components:

$$\begin{matrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ & c_{22} & c_{23} & 0 & 0 & 0 \\ & & c_{22} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{55} & 0 \\ & & & & & c_{55} \end{matrix} \tag{6.1}$$

where $c_{44} = \frac{1}{2}(c_{22} - c_{23})$ and the elasticity tensor is assumed to be positive-definite.

The following Proposition is needed for further analysis:

PROPOSITION 6.1. If the components of the elasticity tensor (6.1) satisfy the relation

$$\begin{aligned} &(c_{55} - 9c_{11})\, c_{12}^{\;4} + 2c_{55}(c_{55} - 17c_{11})c_{12}^{\;3} + \\ &(c_{55}^3 - 45c_{11}c_{55}^2 - 5c_{11}c_{22}c_{55} - 8c_{11}^2 c_{55} + 9c_{11}^2 c_{22})c_{12}^{\;2} + \\ &2c_{11}c_{55}(5c_{11}c_{22} - 12c_{55}^{\;2} - 4c_{11}c_{55} - 5c_{22}c_{55})c_{12} + \\ &c_{11}c_{22}c_{55}(4c_{11}c_{22} - 9c_{55}^{\;2} - 3c_{11}c_{55}) = 0 \end{aligned} \tag{6.2}$$

Then

a) At the parameter $x = \rho c^2$ determined by the polynomial equation

$$\begin{aligned} &c_{11}(c_{11} - c_{55})x^3 + c_{11}(c_{22}c_{55} + 2c_{12}^{\;2} - c_{11}c_{55} - 2c_{11}c_{22})x^2 + \\ &(c_{11}c_{22} - c_{12}^2)(c_{11}c_{22} + 2c_{11}c_{55} - c_{12}^2)x - c_{55}(c_{11}c_{22} - c_{12}^2)^2 = 0 \end{aligned} \tag{6.3}$$

the Jordan normal form $\mathbf{J}_6^{(\mathrm{II})}$ appears in the structure of the matrix $\mathbf{R}_6$;

b) At any other value of the phase speed $c \in (0;\ c_3^{\lim})$, there is no genuine Rayleigh wave admitting the representation (1.1) and propagating on a traction-free boundary of the transversely isotropic half-space.

Proof of Proposition 6.1 can be found in [1].

REMARK 6.1. Equation (6.3) for a transversely isotropic half-space with the elasticity tensor, which does not satisfy Eq. (6.2), was obtained in [3] by application of the three dimensional complex formalism. It can be shown, that Eq. (6.3) has the unique positive root in the interval $\left(0;\ \ \rho\left(c_3^{\lim}\right)^2\right)$.

Combining Eqs. (5.6) with (6.3) and substituting the corresponding values of the elasticity constants and the phase speed into Eqs. (3.10), (4.8), we arrive to

PROPOSITION 6.2. At the conditions (6.2), (6.3) of Proposition 6.1:
a) The eigenvalues γ_k in the representation (3.8) take the form:

$$
\begin{aligned}
\gamma_1 &= -i\left(\frac{c_{11}c_{22}-(c_{11}+c_{55})\rho c^2-2c_{12}c_{55}-c_{12}{}^2}{2c_{11}c_{55}}\right)^{1/2} \\
\gamma_3 &= -i\left(\frac{c_{22}-c_{23}-2\rho c^2}{2c_{55}}\right)^{1/2}
\end{aligned}
\tag{6.4}
$$

b) The corresponding amplitudes $\mathbf{m}_k{}'$ ($\mathbf{m}_1'$ and $\mathbf{m}_3'$ are of the unit length) are of the form:

$$
\begin{aligned}
\mathbf{m}_1' &= p\left(\beta\nu - i\alpha\,\mathbf{n}\right), \\
\mathbf{m}_2' &= sp\left(-i\alpha\nu + \beta\,\mathbf{n}\right), \\
\mathbf{m}_3' &= \nu\times\mathbf{n}
\end{aligned}
\tag{6.5}
$$

where

$$
\alpha = \left(1-\frac{\rho c^2}{c_{55}}\right)^{1/4}, \qquad \beta = \left(\frac{c_{22}-\rho c^2}{c_{11}}\right)^{1/4}
\tag{6.6}
$$

p is the normalization factor:

$$
p = \left(\alpha^2+\beta^2\right)^{-1/2}
\tag{6.7}
$$

and the parameter s is obtained by Eq. (4.8):

$$s = -\frac{c_{11}\beta^2 + c_{55}\alpha^2}{(\alpha\beta)^2 (c_{11} - c_{55}) + (c_{22} - c_{55})} \tag{6.8}$$

c) The partial surface tractions (5.4) are of the form

$$\begin{aligned}
\mathbf{t}_1 &= \left(\left(\gamma_1 c_{11}\beta - i c_{12}\alpha \right) \boldsymbol{\nu} + \left(c_{55} \left(\beta - i\gamma_1\alpha \right) \right) \mathbf{n} \right) e^{ir(\mathbf{n}\cdot\mathbf{x} - ct)} \\
\mathbf{t}_2 &= \left(\left(\left(c_{11} + c_{12}s \right)\beta - i\gamma_1 c_{11} s\alpha \right) \boldsymbol{\nu} + \left(c_{55} \left(\gamma_1 s\beta - i\left(1 - s\right)\alpha \right) \right) \mathbf{n} \right) e^{ir(\mathbf{n}\cdot\mathbf{x} - ct)} \\
\mathbf{t}_3 &= \gamma_3 c_{55} \mathbf{w} e^{ir(\mathbf{n}\cdot\mathbf{x} - ct)}
\end{aligned} \tag{6.9}$$

Substituting the surface tractions (6.9) into the dispersion equation (5.5) yields

COROLLARY. At the conditions of Propositions 6.1, 6.2
a) The dispersion equation (5.5) takes the form

$$\mathbf{t}_1 \times \mathbf{t}_2 \equiv 0, \qquad C_3 = 0 \tag{6.10}$$

b) The nontrivial coefficients C_1, C_2 defined up to arbitrary scalar multiplier by Eq. (5.2), are of the form

$$C_1 = -\frac{\left(\left(c_{11} + c_{12}s \right)\beta - i\gamma_1 c_{11} s\alpha \right)}{\left(\gamma_1 c_{11}\beta - i c_{12}\alpha \right)}, \qquad C_2 = 1 \tag{6.11}$$

Thus, Propositions 6.1 – 6.3 completely characterize the surface wave propagating on a basal plane of the transversely isotropic half space and corresponding to the representation (3.8).

The question, whether there exists a surface wave admitting the representation (3.9), remains open.

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